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Multivariate L-spline Interpolation

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1. INTRODUCTION

In this paper, we generalize the results of [2] and [3] concerning interpolation by multivariate L-splines defined on rectangular partitions. In Section 2, we sharpen the results of [6] for interpolation by "γ-elliptic L-splines", cf. Section 2, of one variable. In Section 4, we define and study the interpolation of smooth, real-valued functions which are defined on a rectangular parallelepiped and in Section 5 we define and study the interpolation of smooth, real-valued functions which are defined on a ball. We remark that the interpolation schemes introduced in this paper may be used, in an obvious manner, to develop multivariate quadrature schemes. The details of this development are left to the reader.

Now we recall some multivariate notation which will be used throughout this paper. For any point

$$x \equiv (x_1, \ldots, x_N) \in R^N, \qquad |x| \equiv (x_1^2 + \cdots + x_N^2)^{1/2}.$$

If $\alpha \equiv (\alpha_1, \ldots, \alpha_N)$ is an N-tuple with nonnegative integer components, then

$$x^\alpha \equiv x_1^{\alpha_1} \ldots x_N^{\alpha_N}, \qquad D^\alpha \equiv D_1^{\alpha_N} \ldots D_N^{\alpha_N} \equiv \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}},$$

$|\alpha| \equiv \alpha_1 + \cdots + \alpha_N$, and $\bar{\alpha} \equiv \max_{1 \leqslant i \leqslant N} \alpha_N$. If Ω is a closed, bounded set in R^N and $n \leqslant m$ are nonnegative integers, then

$$C_n^m(\Omega) \equiv \left\{ f \in C^m(\operatorname{int} \Omega) \,|\, f \text{ is real-valued}, \sum_{|\alpha| \leqslant m} \sup_{x \in \operatorname{int} \Omega} |D^\alpha f(x)| < \infty, \right.$$
$$\left. \text{and } D^\alpha f(x) = 0 \text{ for all } x \in \partial\Omega \text{ and for all } \alpha \text{ with } |\alpha| \leqslant n - 1 \right\}$$

and

$$\|f\|_{C^m(\Omega)} \equiv \sum_{|\alpha| \leqslant m} \sup_{x \in \operatorname{int} \Omega} |D^\alpha f(x)|, \qquad \text{for all } f \in C_n^m(\Omega).$$

Similarly, if t is a positive integer and n is a nonnegative integer with $n \leqslant t$, $B_n^t(\Omega)$ is the set of all real-valued functions $f \in L^2(\Omega)$ such that $D^\alpha f$ exists almost everywhere and is a square integrable function for all α such that $\bar{\alpha} \leqslant t$, $D^\alpha f \in C(\Omega)$ for all α with $|\alpha| \leqslant Nt - 1$, and $D^\alpha f(x) = 0$ for all $x \in \partial\Omega$ and all α with $|\alpha| \leqslant n - 1$. We set

$$\|f\|_{B^t(\Omega)} \equiv (\sum \|D^\alpha f\|^2_{L^2(\Omega)})^{1/2}$$

for all $f \in B_n{}^t(\Omega)$, where the summation is over all α with $\bar{\alpha} \leqslant t$. Moreover, we define

$$B_n^{t,2t}(\Omega) \equiv \{f \in B_n{}^t(\Omega) | D_i{}^k f \in B^t(\Omega),\ 0 \leqslant k \leqslant t,\ 1 \leqslant i \leqslant N\},$$

and

$$\|f\|_{B_n^{t,2t}(\Omega)} \equiv (\sum \|D_i{}^k f\|^2_{B^t(\Omega)})^{1/2},$$

where the summation is over all k, $0 \leqslant k \leqslant t$, and all i, $1 \leqslant i \leqslant N$. Finally, the symbol K will be used repeatedly to denote a positive constant, not necessarily the same at each occurrence.

2. One-dimensional Results

In this section, we introduce γ-elliptic L-splines and study their properties. In particular, we prove analogs of Theorems 6–9 of [6], which give error bounds for interpolation by L-splines.

Given $-\infty < a < b < \infty$, for each nonnegative integer M, let $\mathscr{P}_M$ denote the set of all partitions, Δ, of $[a, b]$, of the form

$$\Delta : a = x^0 < x^1 < \cdots < x^M < x^{M+1} = b,$$

and let $\mathscr{P} \equiv \bigcup_{M=0}^\infty \mathscr{P}_M$. Define $\bar{\Delta} \equiv \max_{0 \leqslant i \leqslant M} (x^{i+1} - x^i)$. Throughout this section, the norms used are the L^2 norm over $[a, b]$ except where explicitly indicated otherwise. If m is a positive integer, let L be any mth-order linear differential operator of the form

$$L(u(x)) \equiv \sum_{j=0}^m a_j(x)\, D^j u(x), \qquad m \geqslant 1, \tag{2.1}$$

where we assume that the coefficient functions $a_j(x) \in B^m(a, b)$ and that there exists a positive real number ω such that

$$a_m(x) \geqslant \omega > 0 \quad \text{for all} \quad x \in [a, b]. \tag{2.2}$$

The formal adjoint of L is given by $L^*(v(x)) \equiv \sum_{j=0}^m (-1)^j D^j(a_j(x) v(x))$. For each $\Delta \in \mathscr{P}$, let $Z_\Delta{}^m$ denote the set of admissible incidence vectors defined as follows: if $\Delta \in \mathscr{P}_0$,

$$Z_\Delta{}^m = \varnothing\,;$$

if $\Delta \in \mathscr{P}_M$, where $M > 1$,

$Z_\Delta{}^m \equiv \{z | z$ is an M-vector with integer components z_i satisfying $1 \leqslant z_i \leqslant m\}$.

For each $\Delta \in \mathscr{P}_N$ and $z \in Z_\Delta{}^m$, $\mathrm{Sp}(L,\Delta,z)$ denotes the collection of all real-valued functions, $s(x)$, called *L-splines*, defined on $[a,b]$ such that

$$L^*L(s(x))=0 \quad \text{for all} \quad x\in(x^j,x^{j+1}), \qquad \text{for each } 0\leqslant j\leqslant M, \tag{2.3}$$

and

$$D^k s(x^j-)=D^k s(x^j+) \quad \text{for all} \quad 0\leqslant k\leqslant 2m-1-z_j, \qquad 1\leqslant j\leqslant M. \tag{2.4}$$

An L-spline, $s(x)$, is said to be a *γ-elliptic L-spline* if and only if

$$\gamma\|D^m w\|\leqslant\|L(w)\| \tag{2.5}$$

for all $w\in B_m{}^m([a,b])$. Finally, we define the interpolation mapping $I: C^{m-1}(a,b)\to \mathrm{Sp}(L,\Delta,z)$, by $I(f)\equiv s(x)$, where

$$D^k s(x^j)\equiv\begin{cases} D^k f(x^j), & 0\leqslant k\leqslant z_j-1, \quad 1\leqslant j\leqslant M,\\ D^k f(x^j), & 0\leqslant k\leqslant m-1, \quad j=0,\, M+1.\end{cases}$$

We remark that this mapping actually corresponds to the type I interpolation of [6] and as such, the *nontrivial* fact that it is well-defined was shown in [6]. Mappings corresponding to the types II, III, and IV interpolation of [6] can be defined, too, for which results analogous to those of this paper are true. The details are left to the reader.

We begin by recalling the "integral relations" of Theorems 4 and 5 of [6].

THEOREM 2.1. *Let L be a differential operator of the form* (2.1) *such that* (2.2) *is satisfied.*

(i) *If $z\in Z_\Delta{}^m$, $\Delta\in\mathscr{P}$, and $f\in B^m(a,b)$, then*

$$\|L(f)\|^2=\|L(f-If)\|^2+\|L(If)\|^2.$$

(ii) *If $z\in Z_\Delta{}^m$, $\Delta\in\mathscr{P}$, and $f\in B^{2m}(a,b)$, then*

$$\|L(f-If)\|^2\leqslant\|f-If\|\,\|L^*L(f)\|.$$

Now we prove improved analogs of Theorems 6–9 of [6].

THEOREM 2.2. *Let L be a differential operator of the form* (2.1) *such that* (2.2) *and* (2.5) *are satisfied.*

(i) *If $0\leqslant j\leqslant m$, $z\in Z_\Delta{}^m$, $\Delta\in\mathscr{P}$, and $f\in B^m(a,b)$, then*

$$\|D^j(f-If)\|\leqslant\frac{m!}{\gamma j!}\left(\frac{\bar\Delta}{\pi}\right)^{m-j}\|L(f-If)\|\leqslant\frac{m!}{\gamma j!}\left(\frac{\bar\Delta}{\pi}\right)^{m-j}\|Lf\|.$$

(ii) *If $0\leqslant j\leqslant m-1$, $z\in Z_\Delta{}^m$, $\Delta\in\mathscr{P}$, and $f\in B^m(a,b)$, then*

$$\begin{aligned}\|D^j(f-If)\|_{L^\infty(a,b)}&\leqslant\frac{m!}{2\gamma j!}\left(\frac{\pi}{j+1}\right)^{1/2}\left(\frac{\bar\Delta}{\pi}\right)^{m-j-1/2}\|L(f-If)\|\\&\leqslant\frac{m!}{2\gamma j!}\left(\frac{\pi}{j+1}\right)^{1/2}\left(\frac{\bar\Delta}{\pi}\right)^{m-j-1/2}\|Lf\|.\end{aligned}$$

Proof. Since $f-If\in C^{m-1}[a,b]$, we can apply to it Rolle's Theorem; thus there exist points $\{\xi_l^{(j)}\}_{l=0}^{M+1-j}$ in $[a,b]$ such that

$$D^j(f-If)(\xi_l^{(j)})=0, \qquad 0\leqslant l\leqslant M+1-j, \qquad 0\leqslant j\leqslant m-1,$$

where

$$a=\xi_0^{(j)}<\xi_1^{(j)}<\cdots<\xi_{M+1-j}^{(j)}=b, \qquad \text{and} \qquad \xi_l^{(j)}\leqslant\xi_l^{(j+1)}\leqslant\xi_{l+1}^{(j)},$$

$0\leqslant l\leqslant M+1-j$, $0\leqslant j\leqslant m-1$. Moreover, $|\xi_{l+1}^{(j)}-\xi_{l+1}^{(j)}|\leqslant(j+1)\bar{\Delta}$ for any $0\leqslant l\leqslant M-j$, $0\leqslant j\leqslant m-1$. Hence, applying the Rayleigh–Ritz inequality, we have

$$\int_{\xi_l^{(j)}}^{\xi_{l+1}^{(j)}}[D^j(f-If)]^2\,dx\leqslant\left[\frac{(j+1)\bar{\Delta}}{\pi}\right]^2\int_{\xi_l^{(j)}}^{\xi_{l+1}^{(j)}}[D^{j+1}(f-If)]^2\,dx, \tag{2.6}$$

for $0\leqslant l\leqslant M-j$, $0\leqslant j\leqslant m-1$. Summing both sides of (2.6) with respect to l, and applying the resulting inequality repeatedly, we obtain

$$\|D^j(f-If)\|\leqslant\frac{m!}{j!}\left(\frac{\bar{\Delta}}{\pi}\right)^{m-j}\|D^m(f-If)\| \tag{2.7}$$

for $0\leqslant j\leqslant m$. The result then follows from (2.5) and (i) of Theorem 2.1.

To prove (ii), it suffices to remark that given $x\in[a,b]$, there exists $\xi_l^{(j)}\in[a,b]$ such that $|x-\xi_l^{(j)}|\leqslant(j+1)\bar{\Delta}$, and

$$D^j(f-If)(x)=\int_{\xi_l^{(j)}}^{x}D^{j+1}(f-If)(t)\,dt.$$

Hence,

$$\|D^j(f-If)\|_{L^\infty(a,b)}\leqslant[(j+1)\bar{\Delta}]^{1/2}\,\|D^{j+1}(f-If)\|,$$

and (ii) follows from (i).

(Q.E.D.)

THEOREM 2.3. *Let L be a differential operator of the form* (2.1) *such that* (2.2) *and* (2.5) *are satisfied.*

(i) *If* $0\leqslant j\leqslant m$, $z\in Z_\Delta{}^m$, $\Delta\in\mathscr{P}$, *and* $f\in B^{2m}(a,b)$, *then*

$$\|D^j(f-If)\|\leqslant\frac{1}{j!}\left(\frac{m!}{\gamma}\right)^2\left(\frac{\bar{\Delta}}{\pi}\right)^{2m-j}\|L^*Lf\|.$$

(ii) *If* $0\leqslant j\leqslant m-1$, $z\in Z_\Delta{}^m$, $\Delta\in\mathscr{P}$, *and* $f\in B^{2m}(a,b)$, *then*

$$\|D^j(f-If)\|_{L^\infty(a,b)}\leqslant\frac{1}{2j!}\left(\frac{m!}{\gamma}\right)^2\left(\frac{\pi}{j+1}\right)^{1/2}\left(\frac{\bar{\Delta}}{\pi}\right)^{2m-j-1/2}\|L^*Lf\|.$$

Proof. Combining (ii) of Theorem 2.1 and (i) of Theorem 2.2, we have

$$\|L(f-If)\| \leqslant \frac{m!}{\gamma}\left(\frac{\bar{\Delta}}{\pi}\right)^m \|L^* Lf\|. \tag{2.8}$$

The result then follows from (i) and (ii) of Theorem 2.2.

(Q.E.D.)

3. Preliminary N-Dimensional Results

In this section, we introduce multivariate γ-elliptic L-splines and study their interpolation properties in rectangular parallelepipeds. In particular, we consider the interpolation of smooth functions, which along with sufficiently many partial derivatives vanish on the boundary of the domain.

For each positive integer i, $1 \leqslant i \leqslant N$, let $-\infty < a_i < b_i < \infty$, let L_i be an mth-order differential operator of the form (2.1) defined on $[a_i, b_i]$ and satisfying (2.2) and (2.5) for some constants ω_i and γ_i, and let $H \equiv \bigtimes_{i=1}^{N} [a_i, b_i]$. Throughout this section, the norm used is the L^2 norm over H. We define the set of partitions of H by

$$\mathscr{P} \equiv \left\{\rho \equiv \bigtimes_{i=1}^{N} \Delta_i \,|\, \Delta_i \in \mathscr{P}(a_i, b_i),\ 1 \leqslant i \leqslant N\right\},$$

and set

$$Z_\rho^m \equiv \left\{z \equiv \bigotimes_{i=1}^{N} z^i \,|\, z^i \in Z^m_{\Delta_i},\ 1 \leqslant i \leqslant N\right\}, \qquad \bar{\rho} \equiv \max_{1 \leqslant i \leqslant N} \bar{\Delta}_i.$$

Moreover, for each $1 \leqslant i \leqslant N$, let I_i denote the interpolation mapping from $C^{m-1}(a_i, b_i)$ to $\mathrm{Sp}(L_i, \Delta_i, z^i)$, for all $z \in Z^m_{\Delta_i}$, $\Delta_i \in \mathscr{P}(a_i, b_i)$, defined in Section 2, and let $I^i \equiv \bigtimes_{j=1}^{i} I_j$. We remark that if $f \in C^{m-1}(H)$, then $I_i(f)$ is interpreted to mean that I_i is applied to f, viewed as a function of the ith variable x_i, with the other variables x_j, $1 \leqslant j \leqslant N$, $j \neq i$, held fixed. As in [5], it is elementary to verify

Lemma 3.1. *$I_i I_j f = I_j I_i f$, $1 \leqslant i, j \leqslant N$, for all $f \in C^{m-1}(H)$.*

Lemma 3.2. *If $i \neq j$, then $D_i(I_j(f)) = I_j(D_i f)$, for all f such that*

$$D_j^k D_i f(x) = D_i D_j^k f(x) \in C(H) \quad \text{for} \quad 0 \leqslant k \leqslant m-1.$$

We now prove a multivariate analogue of Theorems 2.2 and 2.3.

THEOREM 3.1. (i) *If α is such that $\bar{\alpha} \leq m$, $z \in Z_\rho{}^m$, $\rho \in \mathscr{P}$, and $f \in B_m{}^m(H)$, then*

$$\|D^\alpha(f - I^N f)\| \leq \sum_{i=1}^{N} \frac{1}{\gamma_i \alpha_i!} \left(\frac{\bar{\Delta}_i}{\pi}\right)^{m-\alpha_i} \prod_{j=i}^{N-1} \frac{1}{\gamma_j} \left(\frac{b_{j+1} - a_{j+1}}{\pi}\right)^{m-\alpha_{j+1}}$$

$$\times \left\| \left(\mathop{\times}_{j=i}^{N} L_j \right) \tilde{D}_{i-1}^{\alpha} f \right\|,$$

where

$$\tilde{D}_k{}^\alpha \equiv \mathop{\times}_{i=1}^{k} D_i^{\alpha_i} \qquad \textit{for } 1 \leq k \leq N.$$

(ii) *If α is such that $\bar{\alpha} \leq m$, $z \in Z_\rho{}^m$, $\rho \in \mathscr{P}$, and $f \in B_m^{m,2m}(H)$, then*

$$\|D^\alpha(f - I^N f)\| \leq \sum_{i=1}^{N} \frac{1}{\gamma_i \alpha_i!} \left(\frac{\bar{\Delta}_i}{\pi}\right)^{2m-\alpha_i} \prod_{j=i}^{N-1} \frac{1}{\gamma_j} \left(\frac{b_{j+1} - a_{j+1}}{\pi}\right)^{m-\alpha_{j+1}}$$

$$\times \left\| \left(\mathop{\times}_{j=i+1} L_j \right) L_i{}^* L_i \tilde{D}_{i-1}^{\alpha} f \right\|.$$

Proof. We prove only (i), since the proof of (ii) is essentially the same. The proof is by induction on the dimension N and the observation that from Lemma 3.2 we have

$$\begin{aligned}
\|D^\alpha(f - I^N f)\| &\leq \|D_N^{\alpha_N}(\tilde{D}_{N-1}^{\alpha} f - I_N \tilde{D}_{N-1}^{\alpha} f)\| \\
&\quad + \|D_N^{\alpha_N} I_N(\tilde{D}_{N-1}^{\alpha} f - \tilde{D}_{N-1}^{\alpha} I^{N-1} f)\| \\
&\leq \frac{m!}{\gamma_N \alpha_N!} \left(\frac{\bar{\Delta}_N}{\pi}\right)^{m-\alpha_N} \|L_N \tilde{D}_{N-1}^{\alpha} f\| \\
&\quad + \left(\frac{b_N - a_N}{\pi}\right)^{m-\alpha_N} \frac{1}{\gamma_{N-1}} \|\tilde{D}_{N-1}^{\alpha}(L_N f - I^{N-1} L_N f)\|,
\end{aligned}$$

where we have used Theorems 2.1 and 2.2, and the Rayleigh–Ritz inequality. (Q.E.D.)

If $L_i \equiv D_i{}^m$ and $\alpha = 0$, the results of Theorem 3.1 can be greatly simplified.

COROLLARY. *Let $L_i \equiv D_i{}^m$, $1 \leq i \leq N$.*

(i) *If $z \in Z_\rho{}^m$, $\rho \in \mathscr{P}$, and $f \in B_m{}^m(H)$, then*

$$\|f - I^N f\| \leq \sum_{i=1}^{N} \left(\frac{\bar{\Delta}_i}{\pi}\right)^m \prod_{j=i}^{N-1} \left(\frac{b_{j+1} - a_{j+1}}{\pi}\right)^m \left\| \left(\mathop{\times}_{j=i}^{N} D_j{}^m \right) f \right\|.$$

(ii) *If all* $z \in Z_\rho{}^m$, $\rho \in \mathscr{P}$, *and* $f \in B_m^{m,2m}(H)$, *then*

$$\|f - I^N f\| \leqslant \sum_{i=1}^{N} \left(\frac{\bar{\Delta}_i}{\pi}\right)^{2m} \prod_{j=i}^{N-1} \left(\frac{b_{j+1} - a_{j+1}}{\pi}\right)^m \left\| \left(\mathop{\times}_{j=i+1}^{N} D_j{}^m \right) D_i^{2m} f \right\|.$$

Finally, we remark that $C^{mN}(H) \subset B^m(H)$ and $C^{m(N-1)}(H) \subset B^{m,2m}(H)$.

4. N-Dimensional Results in Rectangular Parallelepipeds

In this section, we study the interpolation by γ-elliptic L-splines in a rectangular parallelepiped, H, of functions which do *not* necessarily vanish on the boundary of H. In the proof of Theorem 4.1, fundamental use is made of a form of the Calderòn Extension Theorem, cf. [4], which we recall.

Lemma 4.1. *Let* $H_1 \equiv \mathop{\times}_{i=1}^{N} [a_i{}^1, b_i{}^1]$ *and* $H_2 \equiv \mathop{\times}_{i=1}^{N} [a_i{}^2, b_i{}^2]$ *be two rectangular parallelepipeds in* R^N *such that* $H_1 \subset \operatorname{int} H_2$. *If t is any nonnegative integer, there is a bounded linear extension mapping* $\epsilon: C^t(H_1) \to C_i{}^t(H_2)$ *such that* $\epsilon u(x) = u(x)$, $x \in H_1$, *for all* $u \in C^t(H_1)$.

For each $1 \leqslant i \leqslant N$, let L_i be an mth-order differential operator of the form (2.1) defined on $[a_i{}^2, b_i{}^2]$ such that L_i satisfies (2.2) and (2.5) for some constants ω_i and γ_i. For each $\rho \equiv \mathop{\times}_{i=1}^{N} \Delta_i \in \mathscr{P}(H_1)$, we define a partition ρ of H_2 as follows: if

$$\Delta_i : a_i{}^1 = x_i{}^0 < x_i{}^1 < \cdots < x_i^{M+1} = b_i{}^1,$$

define

$$\begin{aligned} \tilde{\Delta}_i\, a_i{}^2 &= x_i^{-k} < x_i^{-k+1} < \cdots < a_i{}^1 = x_i{}^0 < x_i{}^1 < \cdots < x_i^{M+1} \\ &= b_i{}^1 < x_i^{M+2} < \cdots < x_i^{M+k+1} = b_i{}^2, \end{aligned}$$

where k is chosen so that $\bar{\tilde{\Delta}}_i = \bar{\Delta}_i$, and $\tilde{\rho} \equiv \mathop{\times}_{i=1}^{N} \tilde{\Delta}_i$. Moreover, if $z \equiv \bigotimes_{i=1}^{N} z^i \in Z_\rho{}^m$, where $\rho \in \mathscr{P}(H_1)$, we define $\tilde{z} \equiv \bigotimes_{i=1}^{N} \tilde{z}^i \in Z_{\tilde{\rho}}{}^m$, where $\tilde{z}^i$ is the $M + 2k$ vector $(m, \ldots, m, z_1{}^i, \ldots, z_M{}^i, m, \ldots, m)$.

It is easy to verify that if $f \in C^{m-1}(H_2)$, then

$$I^N \left(\bigotimes_{i=1}^{N} \operatorname{Sp}(L_i, \Delta_i, z^i) \right) f(x) = I^N \left(\bigotimes_{i=1}^{N} \operatorname{Sp}(L_i, \tilde{\Delta}_i, \tilde{z}^i) \right) f(x), \qquad \text{for all } x \in H_1.$$

This is true because for each $1 \leqslant i \leqslant N$, the interpolation over the subintervals $[a_i{}^2, a_i{}^1]$, $[a_i{}^1, b_i{}^1]$, and $[b_i{}^1, b_1{}^2]$ is "local".

Combining this observation with Theorem 3.1, we obtain

Theorem 4.1. *There exists a positive constant, K, such that*

(i) *if* α *satisfies* $\bar{\alpha} \leqslant m$, *if* $z \in Z_\rho{}^m$, $\rho \in \rho(H_1)$, *and* $f \in C^{mN}(H_1)$, *then*

$$\|D^\alpha (f - I^N f)\|_{L^2(H_1)} \leqslant K(\bar{\rho})^{m-\bar{\alpha}} \|f\|_{B^m(H_1)}, \tag{4.1}$$

and

(ii) *if* α *satisfies* $\bar{\alpha} \leqslant m$, *if* $z \in Z_\rho{}^m$, $\rho \in \rho(H_1)$, *and* $f \in C^{m(N+1)}(H_1)$, *then*

$$\|D^\alpha(f - I^N f)\|_{L^2(H_1)} \leqslant K(\bar{\rho})^{2m-\alpha}\|f\|_{B^{m,2m}(H_1)}. \tag{4.2}$$

Proof. By our previous observation it suffices to bound the quantity

$$\left\| D^\alpha \left(\epsilon f - I^N \left(\bigotimes_{i=1}^N \mathrm{Sp}(L_i, \bar{\Delta}_i, \bar{z}^i) \right) \epsilon f \right) \right\|_{L^2(H_2)}, \tag{4.3}$$

where ϵ is the mapping given in Lemma 4.1. Inequalities (4.1) and (4.2) follow directly by applying Theorem 3.1 to bound the quantity given in (4.3).

(Q.E.D.)

5. N-Dimensional Results in Balls

In this section, we study the interpolation by γ-elliptic L-splines in an N-dimensional ball, of functions which do not necessarily vanish on the boundary of the ball. To define the interpolation mapping, fundamental use is made of an extension technique of Lions, cf. ([*1*], p. 218).

Lemma 5.1. *Let* Ω_R *be the ball of radius R with center at the origin. For every nonnegative integer, s, the mapping* θ_s *given by*

$$\theta_s u(x) \equiv \begin{cases} u(x), & \text{if } x \in \Omega_R, \\ \sum_{j=0}^{s+1} \lambda_j u\left(\dfrac{Rx}{|x|} - \dfrac{j}{R(x)}(|x| - R)x\right) & \text{if } x \in \Omega_{2R} - \Omega_R, \end{cases} \tag{5.1}$$

where the constants λ_j *are chosen to satisfy*

$$\sum_{j=1}^{s+1} \lambda_j \left(\frac{-j}{s+1}\right)^l = 1,$$

$0 \leqslant l \leqslant s$, *is a bounded linear mapping of* $C^s(\Omega_R)$ *into* $C^s(\Omega_{2R})$.

If $H \equiv \times_{i=1}^{N} [a_i, b_i]$ is a rectangular parallelepiped such that $\Omega_R \subset \operatorname{int} H \subset \Omega_{2R}$, if $\rho \in \mathscr{P} \equiv \mathscr{P}(H)$, $z \in Z_\rho{}^m$, and for each $1 \leqslant i \leqslant N$, L_i is a differential operator of the form (2.1) on $[a_i, b_i]$ such that (2.2) and (2.5) hold for some constants ω_i and γ_i, then for each $f \in C^s(\Omega_R)$, $m - 1 \leqslant s$, we may define

$$I^N \left(\bigotimes_{i=1}^N \mathrm{Sp}(L_i, \Delta_i, z^i) \right) \theta_s f$$

as an "interpolation" of f. We remark that this "interpolation" can be *explicitly* computed, since the finite number of evaluations of $\theta_s f$ and some of its partial derivatives, required at points of ρ outside Ω_R, can be made explicitly by means of (5.1).

Using Theorem 4.1, we may give the following error bounds for this "interpolation" scheme.

THEOREM 5.1. *There exists a positive constant, K, such that*

(i) *if α satisfies $\bar{\alpha} \leqslant m$, if $z \in Z_\rho{}^m$, $\rho \in \mathscr{P}$, and $f \in C^{mN}(\Omega_R)$, then*

$$\left\| D^\alpha \left(f - I^N \left(\bigotimes_{i=1}^{N} \mathrm{Sp}(L_i, \Delta_i, z^i) \right) \theta_{mN} f \right) \right\|_{L^2(\Omega_R)} \leqslant K(\bar{\rho})^{m-\bar{\alpha}} \|f\|_{B^m(\Omega_R)}; \quad (5.2)$$

and

(ii) *if α satisfies $\bar{\alpha} \leqslant m$, if $z \in Z_\rho{}^m$, $\rho \in \mathscr{P}$, and $f \in C^{m(N+1)}(\Omega_R)$, then*

$$\left\| D^\alpha \left(f - I^N \left(\bigotimes_{i=1}^{N} \mathrm{Sp}(L_i, \Delta_i, z^i) \right) \theta_{m(N+1)} f \right) \right\|_{L^2(\Omega_R)} \leqslant K(\bar{\rho})^{2m-\bar{\alpha}} \|f\|_{B^{m,2m}(\Omega_R)}. \quad (5.3)$$

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